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Quasi-metrics on the function spaces $L_p(0 \leq p \leq \infty)$ for a fuzzy measure

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Abstract

We introduce the function spaces $L_p(0 \leq p \leq \infty)$ and the natural quasi-metrics on them based on the Sugeno integral for a fuzzy neasure. It is proved that for every $0 < p, q < \infty$, $L_p = L_q$ holds.

1 Introduction

We study the metric and quasi-metric structures of the function spaces L_p for a fuzzy measure. The definition of L_p for $0 < p < \infty$ depends on the integration for a fuzzy measure. In fact, many definitions of the fuzzy integrals are proposed such as the Choquet integral[1], Sugeno integral[12, 18], Shilkret integral[17], Imaoka integral[9], pan-integral[14], Lehrer integral(concave-integral)[11], convex-integral[10], and so on. In this note, we shall consider $L_p(0 < p < \infty)$ spaces for the Sugeno integral. In the case where $p = 1, \infty$, the function spaces L_0, L_∞ are defined without utilizing the fuzzy integral.

Definition 1 *[3, 4, 5, 7, 8, 16, 19] Let T be a set. A function $\rho(s,t) : T \times T \to [0, +\infty)$ is called a quasi-metric*
$\iff$

1. $\rho(s,t) \geq 0, \quad \rho(s,t) = 0 \iff s = t, \quad s,t \in T$
2. $\rho(s,t) = \rho(t,s), \quad s,t \in T,$ *and*
3. $\exists K \geq 1 \; ; \rho(s,t) \leq K\left(\rho(s,u) + \rho(u,t)\right), \quad s,t,u \in T.$

In the case where $K = 1$, then ρ is called a metric.

Definition 2 *[1, 2, 12, 15, 18] Let $(X, \mathcal{B}(X))$ be a measure space on a set X, that is $\mathcal{B}(X)$ is a σ-algebra on X. A set function $\mu : \mathcal{B}(X) \to [0, +\infty]$ is called a fuzzy measure*
$\iff$

1. $\mu(\emptyset) = 0,$
2. $A \subset B, A, B \in \mathcal{B}(X) \Rightarrow \mu(A) \leq \mu(B).$

A fuzzy measure μ is called subadditive(or μ is a submeasure)
$\iff$
$\mu(A \cup B) \leq \mu(A) + \mu(B)$ *for every $A, B \in \mathcal{B}(X)$.*

A fuzzy measure μ is called weakly subadditive
$\Longleftrightarrow$
$\exists k \geq 1$; $\mu(A \cup B) \leq \mu(A) + k\mu(B)$ *for every* $A, B \in \mathcal{B}(X)$.

A fuzzy measure μ is said to be continuous from below
$\Longleftrightarrow$
$\mu(A_n) \uparrow \mu(A)$ *for any* A_n, $A \in \mathcal{B}(X)$ *such that* $A_n \uparrow A$.

A fuzzy measure μ is said to be continuous from above
$\Longleftrightarrow$
$\mu(B_n) \downarrow \mu(B)$ *for any* B_n, $B \in \mathcal{B}(X)$ *such that* $B_n \downarrow B$ *with* $\mu(B_1) < +\infty$.

Definition 3 *[12, 18] A set $N \in \mathcal{B}(X)$ is called the strongly null set*
$\Longleftrightarrow$
$\mu(A \cup N) = \mu(A)$ *for every* $A \in \mathcal{B}(X)$.

Lemma 4 *[12, 18] Assume μ is subadditive or weakly subadditive. Then N is a strongly null set if and only if $\mu(N) = 0$.*

A function $f : (X, \mathcal{B}(X)) \to (-\infty, +\infty)$ is called measurable if for every real number r, it holds that $\{f > r\} := \{x \in X \mid f(x) > r\} \in \mathcal{B}(X)$.

Let $\mu : (X, \mathcal{B}(X)) \to [0, +\infty]$ be a fuzzy measure and $f : (X, \mathcal{B}(X)) \to [0, +\infty)$ be a non-negative measurable function. Then the Sugeno integrals[1, 18] of f with respect to μ are defined by

$$(Su) \int_X f d\mu := \sup_{r \geq 0} r \wedge \mu(\{f > r\}).$$

2 The space L_0 of all real measurable functions

Let $(X.\mathcal{B}(X))$ be a measure space and μ is a fuzzy measure on $(X.\mathcal{B}(X))$. Denote by $\mathcal{L}_0$ the set of all real valued measurable functions $f : (X, \mathcal{B}(X)) \to (-\infty, +\infty)$. For $f, g \in \mathcal{L}_0$, we set

$$d_0(f, g) = \inf_{r > 0} \arctan\left\{r + \mu(\{x \in X \mid |f(x) - g(x)| > r\})\right\}.$$

Lemma 5 *Assume μ is weakly subadditive. Then $d_0(f, g)$ is a translation invariant quasi-metric on $\mathcal{L}_0$. Furthermore if μ is subadditive, then $d_0(f, g)$ is a metric.*

Remark that for $h \in \mathcal{L}_0$, $d_0(h, 0) = 0$ if and only if $\mu(|f| > r) = 0$ for every $r > 0$. So that we have the following lemma.

Lemma 6 *Assume μ is weakly subadditive. If $d_0(h, 0) = 0$, then $d_0(f \pm h, g) = d_0(f, g)$ for every $f, g \in \mathcal{L}_0$.*

We set

$$\begin{aligned}
\mathcal{N}_0 &= \{h \in \mathcal{L}_0 \mid \ d_0(h, 0) = 0\}, \text{ and} \\
L_0 &= \mathcal{L}_0 / \mathcal{N}_0.
\end{aligned}$$

By the above lemma, d_0 induces naturally the quasi-metric $\tilde{d}_0$ on L_0 by

$$\tilde{d}_0(f + \mathcal{N}_0, g + \mathcal{N}_0) = d_0(f, g).$$

In the sequel we shall identify the equivalence class $f + \mathcal{N}_0$ with f, and also the quasi-metric $\tilde{d}_0$ with d_0.

3 Sugeno L_p space $L_p(Su)$ $(0 < p < \infty)$

In this section we introduce the L_p space $L_p(Su)$ with respect to the Sugeno integral for $0 < p < \infty$. We shall show that if μ is weakly subadditive, then $L_p(Su)$ is a quasi-metric space. Furthermore for every $0 < p, q < \infty$, we have $L_p(Su) = L_q(Su)$ as a set and the quasi-metrics on these two spaces define the same topology.

Let p be $0 < p < \infty$. For a measurable function $f : (X, \mathcal{B}) \to (-\infty, +\infty)$, we set

$$\begin{aligned}
|f|_p &= \left[\sup_{r \geq 0} r \wedge \mu(|f|^p > r)\right]^{\frac{1}{p}}, \\
\mathcal{L}_p &= \{f \mid \ |f|_p < +\infty\}, \text{ and} \\
\mathcal{O}_p &= \{f \in \mathcal{L}_p \mid \ |f|_p = 0\}.
\end{aligned}$$

Lemma 7 *Assume that μ is a weakly subadditive fuzzy measure. Then we have for $p \geq 1$*

$$|f + g|_p \leq 2k^{\frac{1}{p}} \left(|f|_p + |g|_p\right), \quad f, g \in \mathcal{L}_p,$$

and for $0 < p \leq 1$

$$|f + g|_p \leq (2k)^{\frac{1}{p}} \left(|f|_p + |g|_p\right), \quad f, g \in \mathcal{L}_p.$$

Proof

$$\begin{aligned}
|f+g|_p^p &= \sup_{r\geq 0} r\wedge\mu\left(|f+g|^p>r\right)=\sup_{r\geq 0} r\wedge\mu\left(|f+g|>r^{\frac{1}{p}}\right)\\
&\leq \sup_{r\geq 0} r\wedge\mu\left(\left\{|f|>\frac{1}{2}r^{\frac{1}{p}}\right\}\cup\left\{|g|>\frac{1}{2}r^{\frac{1}{p}}\right\}\right)\\
&\leq \sup_{r\geq 0} r\wedge\left[\mu\left(|f|>\frac{1}{2}r^{\frac{1}{p}}\right)+k\mu\left(|g|>\frac{1}{2}r^{\frac{1}{p}}\right)\right]\\
&\leq \left[\sup_{r\geq 0} r\wedge\mu\left(|f|>\frac{1}{2}r^{\frac{1}{p}}\right)+\sup_{r\geq 0} r\wedge k\mu\left(|g|>\frac{1}{2}r^{\frac{1}{p}}\right)\right]\\
&= \left[\sup_{r\geq 0}\left(2^p\frac{r}{2^p}\right)\wedge\mu\left(|f|^p>\frac{r}{2^p}\right)+\sup_{r\geq 0}\left(2^p\frac{r}{2^p}\right)\wedge k\mu\left(|g|^p>\frac{r}{2^p}\right)\right]\\
&\leq 2^p\left[\sup_{r\geq 0}\left(\frac{r}{2^p}\right)\wedge\mu\left(|f|^p>\frac{r}{2^p}\right)+\sup_{r\geq 0}\left(\frac{r}{2^p}\right)\wedge k\mu\left(|g|^p>\frac{r}{2^p}\right)\right]\\
&\leq 2^p\left[|f|_p^p+k|g|_p^p\right]\leq 2^p k\left[|f|_p^p+|g|_p^p\right],
\end{aligned}$$

where we have used the inequality $a\wedge(b+c)\leq a\wedge b+a\wedge c$. *So that we have for* $p\geq 1$,

$$|f+g|_p\leq 2k^{\frac{1}{p}}\left(|f|_p+|g|_p\right),$$

and for $0<p\leq 1$,

$$|f+g|_p\leq 2k^{\frac{1}{p}}2^{\frac{1}{p}-1}\left(|f|_p+|g|_p\right)=(2k)^{\frac{1}{p}}\left(|f|_p+|g|_p\right).$$

Lemma 8 *We have*

$$|cf|_p\leq \mathrm{Max}\{|c|,1\}|f|_p \quad \textit{for real number } c \textit{ and } f\in\mathcal{L}_p.$$

Proof

$$\begin{aligned}
|cf|_p^p &= \sup_{r\geq 0} r\wedge\mu\left(|cf|^p>r\right)\\
&= \sup_{r\geq 0} r\wedge\mu\left(|f|^p>\frac{r}{|c|^p}\right)\\
&= \sup_{r\geq 0}\left(|c|^p\frac{r}{|c|^p}\right)\wedge\mu\left(|f|^p>\frac{r}{|c|^p}\right).
\end{aligned}$$

If $|c|>1$, *then we have*

$$\sup_{r\geq 0}\left(|c|^p\frac{r}{|c|^p}\right)\wedge\mu\left(|f|^p>\frac{r}{|c|^p}\right)\leq |c|^p\sup_{r\geq 0}\left(\frac{r}{|c|^p}\right)\wedge\mu\left(|f|^p>\frac{r}{|c|^p}\right)=|c|^p|f|_p^p.$$

If $|c|\leq 1$, *then we have*

$$\sup_{r\geq 0}\left(|c|^p\frac{r}{|c|^p}\right)\wedge\mu\left(|f|^p>\frac{r}{|c|^p}\right)\leq\sup_{r\geq 0}\left(\frac{r}{|c|^p}\right)\wedge\mu\left(|f|^p>\frac{r}{|c|^p}\right)=|f|_p^p.$$

So that we have the assertion.

Lemma 9 *Assume that $h\in\mathcal{O}_p$, that is, $|h|_p=0$. Then we have*

$$\mu(|h)>r)=0\quad for\ every\quad r>0.$$

In particular, if μ is continuous from below then $h=0$ μ-almost everywhere, that is

$$\mu(|h|>0)=0.$$

Proof *By the definition of $|\ |_p$ we have the assertion.*

Lemma 10 *Let μ be a weakly subadditive fuzzy measure. Then we have*

$$|f\pm h|_p=|f|_p$$

for every $f\in\mathcal{L}_p$ and $h\in\mathcal{O}_p$.

Proof *By Lemma 4 and Lemma 9, for every $r\geq 0$ it follows that the subset $N(r):=\{|h|>r\}$ is a null set. Let $0<\varepsilon<1$ be arbitrarily fixed. Then we have*

$$\begin{aligned}
\mu(|f\pm h|^p>r) &= \mu\left(|f\pm h|>r^{\frac{1}{p}}\right)\\
&\leq \mu\left((\{|f\pm h|>r^{\frac{1}{p}}\}\cap N(\varepsilon r^{\frac{1}{p}})^c)\cup N(\varepsilon r^{\frac{1}{p}})\right)\\
&= \mu(\{|f\pm h|>r^{\frac{1}{p}}\}\cap N(\varepsilon r^{\frac{1}{p}})^c)\\
&= \mu(|f\pm h|>r^{\frac{1}{p}},|h|\leq\varepsilon r^{\frac{1}{p}})\\
&\leq \mu(|f|>(1-\varepsilon)r^{\frac{1}{p}})\\
&= \mu(|f|^p>(1-\varepsilon)^p r).
\end{aligned}$$

So that we have

$$\begin{aligned}
r\wedge\mu(|f\pm h|^p>r) &\leq r\wedge\mu(|f|^p>(1-\varepsilon)^p r)\\
&\leq \frac{1}{(1-\varepsilon))^p}\left[(1-\varepsilon)^p r\wedge\mu(|f|^p>(1-\varepsilon)^p r)\right]\\
&\leq \frac{1}{(1-\varepsilon))^p}|f|_p^p.
\end{aligned}$$

Taking $\sup_{r\geq 0}$ *in the left hand side, we have*

$$|f\pm h|_p^p\leq\frac{1}{(1-\varepsilon))^p}|f|_p^p.$$

Letting $\varepsilon\downarrow 0$, we have the assertion.

Definition 11 *Let μ be a weakly subadditive fuzzy measure. We set*

$$\begin{aligned} L_p : &= \mathcal{L}_p/\mathcal{O}_p \\ \|f+\mathcal{O}_p\|_p : &= |f|_p \ \ for\ f+\mathcal{O}_p \in L_p. \end{aligned}$$

By Lemma 10, the value $\|f+\mathcal{O}_p\|_p$ does not depend on the choice of the representative f of the equivalence class $f+\mathcal{O}_p$. In the sequel we identify the equivalence class $f+\mathcal{O}_p$ with f and write

$$\|f\|_p = |f+\mathcal{O}_p|_p \ \ for\ f \in L_p.$$

Then $\|f\|_p$ determines a translation invariant quasi-metric on L_p as follows.

Theorem 12 *Let μ be a weakly subadditive fuzzy measure. Then the space $(L_p, \|f\|_p)$ is a linear space. The function $\gamma_p(f,g) := \|f-g\|_p$ is a quasi-metric satisfying :*

1. $\gamma_p(cf,0) \leq \text{Max}\{|c|,1\}\gamma_p(f,0)$ *for a real number c and $f \in L_p$,*

2. *in the case where $p \geq 1$,*

$$\gamma_p(f,g) \leq 2k^{\frac{1}{p}}\left(\gamma_p(f,h)+\gamma_p(h,g)\right) \ \ for\ f,g,h \in L_p,$$

3. *in the case where $0 < p \leq 1$,*

$$\gamma_p(f,g) \leq (2k)^{\frac{1}{p}}\left(\gamma_p(f,h)+\gamma_p(h,g)\right) \ \ for\ f,g,h \in L_p,$$

4. $\gamma_p(f+h,g+h) = \gamma_p(f,g)$ *for $f,g,h \in L_p$ (translation invariance of γ).*

Proof *The assertions 1 and 2 follow from Lemma 8 and Lemma 7. The translation invariance is clear.*

Definition 13 *We say the pair $(L_p, \|f\|_p)$ the Sugeno L_p space and denote it by $L_p(Su)$.*

Remark 14 *$L_p(Su)$ is a topological additive group but not necessarily a topological linear space. The linear topological structure of $L_p(Su)$ shall be studied in [7, 13].*

Theorem 15 *For every $0 < p, q < \infty$, we have $L_p = L_q$. Furthermore γ_p and γ_q determine the same topology.*

Proof *We shall prove $L_1(Su) = L_p(Su)$ set theoretically and topologically. By the definitions, we have*

$$\|f\|_p^p = \sup_{r\geq 0} r \wedge \mu\left(|f|^p > r\right) = \sup_{r \geq 0} r^p \wedge \mu\left(|f| > r\right)$$

and

$$\|f\|_1 = \sup_{r\geq 0} r \wedge \mu\left(|f| > r\right).$$

By the inequality $a^p \wedge b \leq (a \wedge b)^p + a \wedge b$, we have

$$\|f\|_p^p \leq \|f\|_1^p + \|f\|_1,$$

which shows $L_1(Su) \subset L_p(Su)$ and the identity map $i : L_1(Su) \to L_p(Su)$ is continuous.

Conversely by the inequality $(a \wedge b)^p \leq (a^p \wedge b)^p + a^p \wedge b$, we have

$$\|f\|_1^p \leq \|f\|_p^{p^2} + \|f\|_p^p.$$

Remark 16 *$L_1(Su)$ is also realized as a truncated L_∞ space M_∞ and we have $L_1(Su) \neq L_0$ in general. If μ is weakly subadditive , continuous from above and $\mu(X) < +\infty$, then we have $L_1 = L_0$, see [7, 13].*

4 L_∞ space

A measurable function f is called essentially bounded if

$$\exists \alpha \geq 0 \quad ; \quad \mu(|f| > \alpha) = 0.$$

For an essentially bounded function f, we set

$$|f|_\infty = \inf\{\alpha \geq 0 \mid \mu(|f| > \alpha) = 0\}.$$

Proposition 17 *Assume μ is continuous from below. Then the following conditions are wquivalent.*

$|f|_\infty = a$

$\Longleftrightarrow$

(1) $\mu(|f| > a) = 0$ and
(2) for every $b < a, \mu(|f| > b) > 0$.

Proof *($\Leftarrow$) is clear.*
($\Rightarrow$)
Assume $\mu(|f| > a) > 0$. Then we have $\{|f| > a + \frac{1}{n}\} \uparrow \{|f| > a\}$, so that $\mu(\{|f| > a + \frac{1}{n}\}) \uparrow \mu(\{|f| > a\}) > 0$ by the continuity from below. Therefore there exists n_0 such that $\mu(\{|f| > a + \frac{1}{n_0}\}) > 0$, which implies $|f|_\infty \geq a + \frac{1}{n_0} > a$. This contradicts to $|f|_\infty = a$.
If there exists $b < a$ satisfying $\mu(|f| > b) = 0$, then it must be $|f|_\infty leb < a$, which also contradicts to $|f|_\infty = a$.

We set

$$\mathcal{L}_\infty = \{f \mid f \text{ is essentially bounded}\}, \text{ and}$$
$$\mathcal{O}_\infty = \{f \in \mathcal{L}_\infty \mid |f|_\infty = 0\}.$$

Lemma 18 *If $h \in \mathcal{O}_\infty$ then for every $\alpha > 0$ it follows that $\mu(|h| > \alpha) = 0$.*

Lemma 19 *Assume that μ is weakly subadditive. Let $f \in \mathcal{L}_\infty$ and $h \in \mathcal{O}_\infty$. Then we have $|f \pm h|_\infty = |f|_\infty$.*

Proof *Assume $|f|_\infty = a$. Let $\varepsilon > 0$ and $\alpha > a$ be arbitrarily fixed. We have*

$$\{|f \pm h| > \alpha + \varepsilon\} \subset \{|f| > \alpha\} \cup \{|h| > \varepsilon\}.$$

Since $\{|h| > \varepsilon\}$ is a strongly null set, we have

$$\mu(\{|f \pm h| > \alpha + \varepsilon\}) \leq \mu(\{|f| > \alpha\}) = 0.$$

So that we have

$$|f \pm h|_\infty \leq \alpha + \varepsilon$$

for every $\varepsilon > 0$ and $\alpha > a$, which implies $|f \pm h|_\infty \leq a = |f|_\infty$.
Conversely, assume that $|f \pm h|_\infty = b$. Let $\varepsilon > 0$ and $\beta > b$ be arbitrarily fixed. We have

$$\{|f| > \beta + \varepsilon\} \subset \{|f \pm h| > \beta\} \cup \{|h| > \varepsilon\}.$$

Since $\{|h| > \varepsilon\}$ is a strongly null set, we have

$$\mu(\{|f| > \beta + \varepsilon\}) \leq \mu(\{|f \pm h| > \beta\}) = 0.$$

So that we have

$$|f|_\infty \leq \beta + \varepsilon$$

for every $\varepsilon > 0$ and $\beta > b$, which implies $|f|_\infty \leq b = |f \pm h|_\infty$.

Definition 20 *Let μ be a weakly subadditive fuzzy measure. We set*

$$\begin{aligned} L_\infty : &= \mathcal{L}_\infty / \mathcal{O}_\infty \\ \|f + \mathcal{O}_\infty\|_\infty : &= |f|_\infty \quad \textit{for } f + \mathcal{O}_\infty \in L_\infty. \end{aligned}$$

By the preceding Lemma, the value $\|f + \mathcal{O}_\infty\|_\infty$ does not depend on the choice of the representative f of the equivalence class $f + \mathcal{O}_\infty$. In the sequel we identify the equivalence class $f + \mathcal{O}_\infty$ with f and write

$$\|f\|_\infty = |f + \mathcal{O}_\infty|_\infty \quad \textit{for } f \in L_\infty.$$

Then $\|f\|_\infty$ determines a norm on L_∞.

Theorem 21 $\|f\|_\infty$ *is a norm on L_∞.*

Proof *For every $\varepsilon > 0$, we have*

$$\mu(|f| > \|f\|_\infty + \varepsilon) = 0, \quad \mu(|g| > \|g\|_\infty + \varepsilon) = 0.$$

Since $\{|f+g| > \|f\|_\infty + \|g\|_\infty + 2\varepsilon\} \subset \{|f| > \|f\|_\infty + \varepsilon\} \cup \{|g| > \|g\|_\infty + \varepsilon\}$, by the weak subsdditivity of μ, we have

$$\mu(|f+g| > \|f\|_\infty + \|g\|_\infty + 2\varepsilon) \leq \mu(|f| > \|f\|_\infty + \varepsilon) + k\mu(|g| > \|g\|_\infty + \varepsilon) = 0,$$

which implies

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty + 2\varepsilon.$$

Letting $\varepsilon \downarrow 0$ we have the triangle inequality.
We show $\|cf\|_\infty = |c| \cdot \|f\|_\infty$. Remark that $\mu(|f| > \alpha) = 0$ if and only if $\mu(|cf| > |c|\alpha) = 0$. This implies $\{|c|\alpha \mid \mu(|f| > \alpha) = 0\} = \{\beta \mid \mu(|cf| > \beta) = 0\}$. Consequently it follows that

$$|c| \cdot \|f\|_\infty = \inf\{|c|\alpha \mid \mu(|f| > \alpha) = 0\} = \inf\{\beta \mid \mu(|cf| > \beta) = 0\} = \|cf\|_\infty.$$

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